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LETTER TO THE EDITOR

Amplitude–exponent relation for the correlation length in the spherical model

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Abstract. For the critical spherical model in $d=r+1$ dimensions, finite-size calculations of the spin–spin and energy–energy correlation lengths suggest relations among the critical exponents and the finite-size scaling amplitudes of the correlation lengths similar to those obtained in $(1+1)$ dimensions from conformal invariance. For antiperiodic boundary conditions, the results agree with numerical studies in the $(2+1)$-dimensional Ising model.

For two-dimensional systems defined on an infinitely long strip of finite width N and periodic boundary conditions, it has been shown that the hypothesis of conformal invariance at the critical point yields a relation between the finite-size scaling amplitude A_i of the inverse correlation length $\xi_i^{-1}=A_iN^{-1}$ and the bulk critical exponent x_i (Cardy 1984, von Gehlen *et al* 1986)

$$A_i=2\pi x_i. \tag{1}$$

One may ask whether a linear amplitude–exponent relation like (1) is possible for three-dimensional models. In a recent letter (Henkel 1987), studying the critical $(2+1)$-dimensional Ising model with a geometry infinite in one direction but finite in the other two directions with antiperiodic boundary conditions, numerical evidence was given for a linear relation $A_i \sim x_i$, where x_i is a bulk critical exponent. Here, we shall show that this linear relationship is also valid for the spherical model.

Consider the $(d=r+1)$-dimensional mean spherical model (for reviews see Joyce 1972, Baxter 1982). In various geometries, the finite-size behaviour of the thermodynamic quantities has been studied for various boundary conditions (Barber and Fisher 1973, Brézin 1982, Luck 1985, Singh and Pathria 1985a, b, 1987). We shall use the Hamiltonian formulation (Henkel and Hoeger 1984) which allows a direct study of several correlation lengths, e.g. the spin–spin and energy–energy correlation lengths. Taking the Hamiltonian limit of the spherical model, we have a geometry which is infinite in one direction but finite in the other directions.

The Hamiltonian in $d=r+1$ dimensions (Henkel and Hoeger 1984) is

$$H=-\tfrac{1}{2}\Delta+\tfrac{1}{2}(\chi\lambda r \boldsymbol{x}^2-\tfrac{1}{2}\lambda \boldsymbol{x}M\boldsymbol{x}) \tag{2}$$

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where Δ is the Laplacian, $\lambda = 2/T^2$ and T is the temperature, M is the interaction matrix and χ is determined from the spherical constraint

$$\tilde{N} = \langle 0|x^2|0\rangle \tag{3}$$

where $\tilde{N}$ is the number of sites in the r-dimensional quantum plane.

Taking a hypercubic lattice with $\tilde{N} = N^r$ and antiperiodic boundary conditions, the eigenvalues μ_k of the matrix M are for nearest-neighbour interactions ($\boldsymbol{k}$ is a collective symbol for the set of numbers $\{k_j;\, j=1,\ldots,r\}$) (Barber and Fisher 1973)

$$\mu_k = 2\sum_{j=1}^{r}\cos(2\pi/N)(k_j+\tfrac{1}{2}) \qquad k_j = 0, 1, \ldots, N-1. \tag{4}$$

H can be diagonalised in terms of free bosonic oscillators a_k:

$$H = \lambda^{1/2}\sum_k\left(\chi r - \sum_{j=1}^{r}\cos(2\pi/N)(k_j+\tfrac{1}{2})\right)^{1/2}(a_k^+a_k+\tfrac{1}{2}). \tag{5}$$

The inverse correlation lengths ξ_i^{-1} are proportional to the energy differences $E_i - E_0$. In particular, the lowest energy gap gives the correlation length of the spin-spin correlation function.

We now take $T = T_c$ and let $N \to \infty$. Then the leading term in $1/N$ in the Hamiltonian is

$$H = \lambda_c^{1/2}\sum_k\left((\chi-1)r + \tfrac{1}{2}(2\pi/N)^2\sum_{j=1}^{r}(k_j+\tfrac{1}{2})^2\right)^{1/2}a_k^+a_k \tag{6}$$

and we have also chosen the zero point of energy such that $E_0 = 0$. Define the 'thermogeometric parameter' (see Pathria 1983)

$$y = (1/\sqrt{2})N[(\chi-1)r]^{1/2} \tag{7}$$

and we have

$$H = \frac{1}{T_c}\frac{2\pi}{N}\sum_k\left(\frac{y^2}{\pi^2} + \sum_{j=1}^{r}(k_j+\tfrac{1}{2})^2\right)^{1/2}a_k^+a_k \tag{8}$$

where y is determined from the constraint equation. From (3) we have

$$\tilde{N} = \tfrac{1}{2}\lambda^{-1/2}\sum_k\left(\chi r - \tfrac{1}{2}\mu_k\right)^{-1/2} \tag{9}$$

$$= \frac{\lambda^{-1/2}}{\sqrt{\pi}}\int_0^\infty du\sum_k \exp\left[-\left(\chi r - \sum_{j=1}^{r}\cos(2\pi/N)(k_j+\tfrac{1}{2})\right)u^2\right]. \tag{10}$$

Using the identity (Singh and Pathria (1985b), equation (27))

$$\sum_{k=0}^{N-1}\exp\{u\cos[(2\pi/N)(k+\tfrac{1}{2})]\} = N\sum_{q=-\infty}^{\infty}\cos(\pi q)I_{Nq}(u) \tag{11}$$

where $I_\nu(u)$ is a modified Bessel function, we arrive at

$$\frac{(2\pi)^{1/2}}{T} = \int_0^\infty du\,\exp(-\chi r u^2)[I_0(u^2)]^r + \sum_q{}'\int_0^\infty du\,\exp(-\chi r u^2)(-1)^{\Sigma q_j}\prod_{j=1}^{r}I_{Nq_j}(u^2) \tag{12}$$

where the prime means that terms with $\boldsymbol{q} = 0$ are excluded. The first term in (12) is the bulk contribution. Close to the critical point $T = T_c$, one has $\chi \approx 1$ and the expansion of the first term around $\chi = 1$ is known (Henkel and Hoeger 1984). For the second term, following Singh and Pathria (1985a, b), we use

$$I_\nu(u) = (2\pi u)^{-1/2} \exp(u - \nu^2/u)(1 + O(u^{-1}))$$
$$\int_0^\infty \mathrm{d}u\, u^{\nu-1} \exp(-\alpha u - \beta/u) = 2(\beta/\alpha)^{\nu/2} K_\nu(2\surd(\alpha\beta)) \tag{13}$$

where $K_\nu(u)$ is the other modified Bessel function. We obtain, to leading order in $1/N$,

$$(2\pi)^{1/2}(1/T - 1/T_c)$$
$$= (2\pi^r)^{-1/2}(y/N)^{r-1}\left(\Gamma\left(\frac{1-r}{2}\right) + \sum_{\boldsymbol{q}}{}' (-1)^{\Sigma q_i}(qy)^{-(r-1)/2} K_{(r-1)/2}(2qy)\right) \tag{14}$$

where $q = (q_1^2 + \ldots + q_r^2)^{1/2}$. This constraint equation for the Hamiltonian limit is completely analogous to the result for the (mean) spherical model (Singh and Pathria (1985b), equation (35)). One can proceed and re-obtain the universality of the finite-size scaling functions in the Hamiltonian limit, in complete analogy with Singh and Pathria (1985a, b, 1987).

From (14), y can be determined numerically. For example, in $d = 3$ ($r = 2$) one has for $T = T_c$, $y = 0.1173\ldots$.

We are interested in the correlation lengths. As mentioned above, the lowest energy of H (equation (8)) is proportional to the inverse spin-spin correlation length. The lowest energy is obtained by a single excitation with all $k_j = 0$ (or -1). The amplitude A_σ of the spin-spin correlation length $\xi_\sigma^{-1} = A_\sigma N^{-1}$ is

$$A_\sigma = \frac{2\pi}{T_c}\left(\frac{y^2}{\pi^2} + r\frac{1}{4}\right)^{1/2}. \tag{15}$$

The corresponding energy-energy correlation length ξ_ε is obtained from two excitations of this type, such that the total 'momentum' of this state, measured by $\boldsymbol{k}_1 + \boldsymbol{k}_2$, vanishes. This definition of ξ_ε is analogous to the definition used in the $(2+1)$-dimensional Ising model (see Henkel 1987). Thus

$$A_\varepsilon = 2A_\sigma. \tag{16}$$

We want to compare this with the bulk critical exponents x_σ and x_ε, defined by (at $T = T_c$)

$$\Gamma_\sigma(\rho) \sim \rho^{-2x_\sigma} \qquad \Gamma_\varepsilon(\rho) \sim \rho^{-2x_\varepsilon} \tag{17}$$

where the spin-spin correlation function Γ_σ and the energy-energy correlation function Γ_ε are given by

$$\Gamma_\sigma = \langle 0|x_1 x_n|0\rangle - \langle 0|x_1|0\rangle^2$$
$$\Gamma_\varepsilon = \langle 0|x_1 x_2 x_n x_{n+1}|0\rangle - \langle 0|x_1 x_2|0\rangle^2. \tag{18}$$

The matrix elements can be readily evaluated and yield, using translational invariance and the fact that $\langle 0|x_1|0\rangle = 0$ at $T = T_c$,

$$\Gamma_\varepsilon = \langle 0|x_1 x_n|0\rangle\langle 0|x_2 x_{n+1}|0\rangle + \langle 0|x_1 x_2|0\rangle\langle 0|x_n x_{n+1}|0\rangle - \langle 0|x_1 x_2|0\rangle^2$$
$$= \Gamma_\sigma^2. \tag{19}$$

For example, for $d=3$, one has, from $x_\sigma = \frac{1}{2}(d-2+\eta)$ and the fact that $\eta=0$ in the spherical model, that $x_\sigma = \frac{1}{2}$, $x_\varepsilon = 1$.

Comparing (16) and (19), we find that indeed

$$A_\varepsilon / A_\sigma = x_\varepsilon / x_\sigma. \tag{20}$$

We note that the spherical model has the peculiarity that this relation is satisfied for antiperiodic as well as periodic boundary conditions, for dimensions $2<d<4$. Since the spherical model Hamiltonian (8) is given by a set of free bosonic oscillators, one expects in fact a simple addition of the one-particle energies (see (16)) and also a factorisation of Γ_ε into Γ_σ (see (19)). In contrast, for the (2+1)-dimensional Ising model, equation (20) is not satisfied for periodic boundary conditions, but is satisfied for antiperiodic boundary conditions (Henkel 1987).

We conclude that the calculation presented supports the conjecture that at the critical point the relation $A_i \sim x_i$ also holds for three-dimensional models, in the geometry of one infinite and two finite directions and with antiperiodic boundary conditions.

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